# The generalized inverse Gaussian distribution

A. E. Koudou (Univ. de Lorraine)

University of Luxembourg, March 8, 2018

# Outline

The generalized inverse Gaussian distribution

Some examples of use of the GIG distribution

A few words on Stein's method

Stein characterization of the GIG distribution

Possible applications

- GIG distribution as limit of a triangular scheme.
- GIG as the law of continued fractions
- Approximation of GIG$(p, a, b)$ by a gamma law for $p > 0$ and small $b$

# 1. The generalized inverse Gaussian distribution

The *generalized inverse Gaussian* (hereafter GIG) distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ has density

$$f_{p,a,b}(x) := c(p,a,b)x^{p-1}e^{-(ax+b/x)/2}, \quad x > 0,$$

$$c(p,a,b) := \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})},$$

with $K_p$ the modified Bessel function of the third kind.

$$K_p(z) = 2^{-p-1}z^p \int_0^\infty x^{-p-1}e^{-x-\frac{z^2}{4x}}dx, \quad \mathrm{Re}(z) > 0.$$

If $p < 0$, then $\mathrm{GIG}(p,a,b)$ converges to a gamma distribution as $b \to 0$.
If $p > 0$, then $\mathrm{GIG}(p,a,b)$ converges to a reciprocal gamma distribution as $a \to 0$.

- Introducing $\theta = \sqrt{ab}$ and $\eta = \sqrt{a/b}$, the GIG density can be written
$$\frac{\eta^p}{2K_p(\theta)} x^{p-1} e^{-\frac{1}{2}\theta(\eta x + \eta^{-1} x^{-1})} \quad (x > 0),$$
and one sees that $\theta$ is a concentration parameter and $\eta$ a scale parameter. For some statistical properties of the GIG distribution and applications to real data, see Jorgensen (1982).
- The name "generalized inverse Gaussian" was proposed by Good (1953) in his study of population frequencies. But this three-parameter law appears in a work by Halphen (1941), not signed under his own name, probably because of the war context. For this reason the GIG laws is also called "Halphen type A distribution".

# Some properties of the GIG distribution

Barndorff-Nielsen and Halgreen (1977):

- If $X \sim \mathrm{GIG}(p, a, b)$ then $\frac{1}{X} \sim \mathrm{GIG}(-p, b, a)$.
- For $p,\ a,\ b > 0$, if $X \sim \mathrm{GIG}(-p, a, b)$ and $Y \sim \gamma(p, \frac{a}{2})$, then $X + Y \sim \mathrm{GIG}(p, a, b)$,
  where $\gamma(p, c) = \mathrm{GIG}(p, 2c, 0)$ is the gamma distribution with density
  $$\frac{c^p}{\Gamma(p)} x^{p-1} \exp -cx \quad (x > 0).$$
- $\mathrm{GIG}(-p, b, a)$ is infinitely divisible.
- depending on the sign of $p$, $\mathrm{GIG}(p, a, b)$ can be viewed as the distributions of either first or last exit times of certain diffusion processes. The general case was treated by Vallois (1991) considering Bessel processes with drift.

# A few words about the case $p = -\frac{1}{2}$

If $p = -\frac{1}{2}$, then $GIG(p; a, b)$ is the *inverse Gaussian (IG) distribution* :

$$IG(a,b)(dx) = \sqrt{\frac{b}{2\pi}} e^{\sqrt{ab}} x^{-\frac{3}{2}} e^{-\frac{1}{2}(ax+b/x)} \mathbf{1}_{(0,\infty)}(x) dx.$$

- The history of IG dates back to 1915 when Schrödinger and Smoluchowski obtained, independently, the density of the first passage time of Brownian motion with positive drift. The drift-free case had already been treated by Bachelier (1900) in his thesis on the theory of speculation.
- This distribution was named "inverse Gaussian" by Tweedie (1945), who observed that the cumulant function of this law is the inverse of the cumulant function of the normal law.

- The inverse Gaussian distribution is used in data analysis when the observations are highly right-skewed, e.g. in cardiology, demography, finance, biology, hydrology, pharmacokinetics. See Chhikara and Folks (1989), Seshadri (1999).
- If $\mu = \frac{1}{2}$ we have the reciprocal inverse Gaussian distribution

$$\mathrm{RIG}(a, b)(dx) = \sqrt{\frac{a}{2\pi}} e^{\sqrt{ab}} x^{-\frac{1}{2}} e^{-\frac{1}{2}(ax + b/x)} \mathbf{1}_{(0,\infty)}(x) dx.$$

- IG and RIG laws are respectively the distribution of the first and the last hitting time for a Brownian motion (cf e.g. Bhattacharya and Waymire, 1990).
  For a standard Brownian motion $B$, define, for $a \geq 0$ and $b > 0$,
  $$\tau_b^a = \inf\{t > 0;\ B_t + \sqrt{b}t = a\},$$
  $$\sigma_b^a = \sup\{t > 0;\ B_t + \sqrt{b}t = a\}.$$
  Then $\tau_b^a \sim \mathrm{IG}(a, b)$ and $\sigma_b^a \sim \mathrm{RIG}(a, b)$.
- More generally, depending on the sign of $p$, GIG$(p, a, b)$ can be viewed as the distributions of either first or last exit times of certain diffusion processes (see Barndorff-Nielsen *et al* (1978) and also Vallois (1991) considering Bessel processes with drift).

- The GIG density can be defined on the set of positive definite matrices, the case $a = 0$ defining Wishart matrices (see Letac and Wesolowski, 2000).
- The Black-Scholes formula in finance can be expressed in terms of the distribution function of GIG variables (see Madan, Roynette and Yor, 2008).

**The Matsumoto-Yor property**. *Let $p > 0$, $a > 0$ and $b > 0$. Consider two independent, non-Dirac, positive random variables $X$ and $Y$ such that*

$$X \sim \mathrm{GIG}(-p, a, b), \quad Y \sim \gamma(p, a/2) = \mathrm{GIG}(p, a, 0).$$

*Then the random variables*

$$U = \frac{1}{X+Y}, \quad V = \frac{1}{X} - \frac{1}{X+Y}$$

*are independent if and only if there exist $p > 0$, $a > 0$ and $b > 0$ such that $X \sim \mathrm{GIG}(-p, a, b)$ while $Y \sim \gamma(p, a/2)$.*
For a proof see Letac and Wesołowski (2000), Matsumoto and Yor (2001).

## Some examples of use of the GIG distribution

- Jorgensen (1982) proved a better fit of GIG than the exponential distribution to data consisting in
  - intervals (in hours) between successive failures of airconditioning equipment in Boeing 720 aircraft;
  - intervals between pulses along a nerve fibre;
  - intervals between the times at which vehicles pass a point on a road.
- Iyengar & Liao (1997) : neural activity (interspike intervals for neurons); comparison of the GIG fit with the lognormal fit.
- Chebana *et al* (2010) : application to extreme hydrologic events

## A few words on Stein's method

- Stein (1974) derived a technique to obtain bounds for normal approximation.
- This technique relies on the fact that a rv $W$ follows the $\mathrm{N}(0, \sigma^2)$ distribution if and only if

$$\mathbb{E}(\sigma^2 f'(W) - Wf(W)) = 0$$

  for regular functions $f$.
- This leads to the so-called Stein equation

$$\sigma^2 f'(x) - xf(x) = h(x) - Nh$$

  where $Nh = \mathbb{E}(h(Z))$ for $Z \sim N(0, \sigma^2)$.
- To find a bound for $|\mathbb{E}(h(W)) - Nh|$ for a rv $W$, it is enough to bound $|\mathbb{E}(\sigma^2 f'(W) - Wf(W))|$ for $f$ solution of the above Stein equation.
- The method has been extended to several other distributions over the years.

# Stein characterization of GIG distributions

## Proposition

*A random $X$ follows the GIG$(p, a, b)$ distribution if and only if, for any regular function $f$ ,*

$$\mathbb{E}\left[2X^2(f'(X)) + (-aX^2 + 2(p+1)X + b)f(X)\right] = 0.$$

The proof (see K. and Ley (2014) readily comes from the general result by Ley and Swan (2013) that a random variable $X$ follows a distribution with density $h$ if and only if, for a suitable class of functions $f$,

$$\mathbb{E}\left[f'(X) + \frac{h'(X)}{h(X)}f(X)\right] = 0,$$

the result follows by replacing $h$ with the density of GIG$(p, a, b)$.

This result was also established by Gaunt (2017) as a special case of his Stein characterization of the generalized hyperbolic distribution. Moreover, he solved the GIG Stein equation using a proposition of Schoutens (2001), and bounded the solution thanks to a lemma of Schoutens (1999). Namely, he found that the GIG Stein equation

$$2x^2 f'(x)+(-aX^2+2(p+1)x+b)f(x) = h(x)-\mathbb{E}(h(X)),\ X \sim GIG(p,a,b)$$

has solution

$$f(x) = -\frac{1}{2x^2 p(x)} \int_x^\infty (h(t) - \mathbb{E}(h(X)))p(t)dt,$$

where $p$ is the density function of $GIG(p,a,b)$. Denoting $l = \mathbb{E}(X)$ he found that

$$\|f\| \leq \frac{1}{2l^2 p(l)} \|h - \mathbb{E}(h(X))\|,$$

thus $f$ is bounded if $h$ is bounded.

# Possible applications

a) GIG distribution as limit of a triangular scheme. The family of generalized $\Gamma$-convolutions (see Thorin, 1977) consists of distributions on $[0, \infty)$ whose characteristic function has the form

$$\phi(u) = \exp\left[iu\alpha - \int_0^\infty \log\left(1 - \frac{iu}{y}\right) dU(y)\right]$$

for some $\alpha \geq 0$ and some non-decreasing function $U : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$U(0) = 0, \int_0^1 |\log y| dU(y) < \infty, \quad \int_1^\infty \frac{1}{y} dU(y) < \infty.$$

The GIG$(p, a, b)$ distribution belongs to the family of generalized $\Gamma$-convolutions (see for instance Eberlein and Hammerstein, 2004), with

$$\alpha = 0,\ U(x) = \left[\max(0, p) + b \int_{a/2}^{x} g_{|p|}(2by - ab)\, dy\right] \mathbb{I}_{(a/2,\infty)}(x),$$

with

$$g_\nu(x) = \frac{2}{\pi^2 x [J_\nu^2(\sqrt{x}) + Y_\nu^2(\sqrt{x})]},\ x > 0,$$

where $J_\nu$ and $Y_\nu$ denote the Bessel functions of first and second kind with index $\nu$.

## Theorem

*(Eberlein and Hammerstein, 2004). Consider an arbitrary* $GIG(p, a, b)$ *distribution with corresponding function* $U$ *defined above.*
*Consider a sequence* $(K_n)_{n\geq 1}$ *such that* $K_n > a/2$*. For each* $n \geq 1$*, consider a partition*

$$a/2 = x_{n,1} < x_{n,2} < \cdots < x_{n,k_n} = K_n$$

*of* $[a/2, K_n]$*.*
*Let* $(X_{n,i}, 1 \leq i \leq k_n, n \geq 1)$ *be a family of independent random variables such that*

- $X_{n,1} \sim \gamma(p, a/2)$ *if* $p > 0$ *or* $X_{n,1} = 0$ *otherwise.*
- *For* $2 \leq i \leq k_n$*,* $X_{n,1} \sim \gamma(x_{n,i}, U(x_{n,i}) - U(x_{n,i-1}))$*.*

*Then, if* $K_n \to \infty$ *and* $\sup_{1\leq i\leq k_n} |x_{n,i}) - x_{n,i-1}|$ *tends to 0 as* $n \to \infty$*, then* $\mathcal{L}(\sum_{i=1}^{K_n} X_{n,i})$ *converges to* $GIG(p, a, b)$*.*

If $(\bar{X}_{n,i}, 1 \leq i \leq k_n, n \geq 1)$ is a family of rowwise iid random variables such that

$$\mathcal{L}(\bar{X}_{n,i}) = \mathcal{L}\left(\frac{1}{n}\sum_{i=1}^{K_n} X_{n,i}\right),$$

then the process

$$(S_n(t) := \sum_{i=1}^{[nt]} \bar{X}_{n,i},\ 0 \leq t \leq 1)$$

converges in law to the GIG Lévy process $(\tau(t), 0 \leq t \leq 1)$.
Question : Use Stein's method to derive bounds for the distance between $\mathcal{L}(\sum_{i=1}^{K_n} X_{n,i})$ and GIG$(p, a, b)$.

## b) GIG distribution as the law of a continued fraction.

**Theorem**

*Letac-Seshadri (1983).*

- *Let $X$ and $Y$ be two independent random variables such that $X > 0$ and $Y \sim \gamma(p, a/2)$ for $p, a > 0$. Then $X =_d \frac{1}{Y+X}$ if and only if $X \sim \mathrm{GIG}(-p, a, a)$.*
- *Let $X$, $Y_1$ and $Y_2$ be three independent random variables such that $X > 0$, $Y_1 \sim \gamma(p, b/2)$ and $Y_2 \sim \gamma(p, a/2)$ for $p, a, b > 0$. Then $X =_d \frac{1}{Y_1 + \frac{1}{Y_2+X}}$ if and only if $X \sim \mathrm{GIG}(-p, a, b)$.*

If $(Y_i)_{i\geq 1}$ is a sequence of independent random variables such that

$$\mathcal{L}(Y_{2i-1}) = \mathcal{L}(Y_1) = \gamma(\lambda, b/2) \textit{ and } \mathcal{L}(Y_{2i}) = \mathcal{L}(Y_2) = \gamma(\lambda, a/2); \;\; i \geq 1,$$

then

$$\mathcal{L}\left(\cfrac{1}{Y_1 + \cfrac{1}{Y_2 + \cfrac{1}{Y_3 + \ddots}}}\right) = GIG(-\lambda, a, b).$$

Question : Use Stein's method to derive bounds for the distance between

$$\mathcal{L}\left(\frac{1}{Y_1 + \frac{1}{Y_2 + \frac{1}{\cdots + \frac{1}{Y_n}}}}\right)$$

and $\mathrm{GIG}(p, a, b)$.

c)Approximation of GIG$(p, a, b)$ by the gamma law $\gamma(p, a/2)$ for $p > 0$ and small $b$.

As recalled earlier, if $p > 0$, then GIG$(p, a, b)$ converges to $\gamma(p, a/2)$ as $b \to 0$.

Question : Use Stein's method to derive bounds for the distance between GIG$(p, a, \frac{1}{n})$ and $\gamma(p, a/2)$

# References


- Barndorff-Nielsen, O. E. and Halgreen, C. (1977). Infinite divisibility of the Hyperbolic and generalized inverse Gaussian distribution. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **38**, 309–312.
- Bhattacharya, R. N. and Waymire, E. C. (1990). *Stochastic Processes with Applications*. New York: Wiley.
- Chebana, F., El Adlouni, S. and Bobée, B. (2010). Mixed estimation methods for Halphen distributons with applications in extreme hydrologic events. *Stoch. Environ. Res. Risk Assess.*, **24**, 359–376.
- Gaunt, R. E. (2017). A Stein characterization of the generalized hyperbolic distribution. ArXiv .

- Good, I. J. (1953).The population frequencies of species and the estimation of population parameters, *Biometrika*, **40**, 237–260.
- Iyengar, S. and Liao, Q. (1997). Modeling neural activity using the generalized inverse Gaussian distribution. *Biol. Cybern.* , **77**, 289–295.
- Jorgensen B. (1982) *Statistical Properties of the Generalized Inverse Gaussian Distribution.* Springer-Verlag, Heidelberg.
- Koudou, A. E. and Ley, C. (2014). Characterizations of GIG laws: a survey. Probab. Surv. **11**, 161–176.

Letac, G. and Wesołowski, J. (2000). An independence property for the product of GIG and gamma laws. *Ann. Prob.* **28**, 1371–1383.

Madan, D., Roynette, B. and Yor, M. (2008). Unifying Black-Scholes type formulae which involve Brownian last passage times up to a finite horizon. *Prépublications de l'Institut Elie Cartan*, 2008/28.

Matsumoto, H. and Yor, M. (2001). An analogue of Pitman's $2M - X$ theorem for exponential Wiener functional, Part II: the role of the generalized inverse Gaussian laws. *Nagoya Math. J.* **162**, 65–86.

Vallois, P. (1991). La loi Gaussienne inverse généralisée comme premier ou dernier temps de passage de diffusion. *Bull. Sc. Math.*, 2^e^ Série, **115**, 301–368.